SPECTRAL REPRESENTATION

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Lecture slides for EEE 554

Arizona State University

Spring 2017

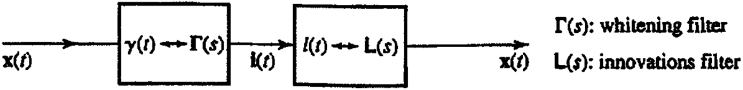
Factorization and Innovation

Continuous Time Processes

Let a WSS process $x(t)$ be given.

Problem: Find a *minimum-phase* system $L(s)$ such that $x(t)$ is the output of the system if a *white noise* $i(t)$ is the input.

> A system $L(s)$ is said to be *minimum-phase* if
> - it is causal and its impulse response $l(t)$ has finite energy ($\int_0^\infty l^2(t)dt < \infty$);
> - its inverse $\Gamma(s) = \frac{1}{L(s)}$ is causal and its impulse response has finite energy.
>
> In other words, $L(s)$ and $\Gamma(s)$ must both be analytic in the right-half-plane.



A process $x(t)$ with such a representation is called *regular* and the process $i(t)$ is called its *innovations.*

Continuous Time Processes

While a solution for $L(s)$ doesn't always exist, it is *unique* when it does. Why? Because with such a representation we have

$$i(t) = \int_0^\infty \gamma(\alpha)x(t-\alpha)d\alpha, \quad R_{ii}(\tau) = \delta_\tau$$

$$x(t) = \int_0^\infty l(\alpha)i(t-\alpha)d\alpha, \quad E\{x^2(t)\} = \int_0^\infty l^2(t)dt < \infty$$

which means

$$S(s) = L(s)L(-s), \quad S(\omega) = |L(j\omega)|^2$$

which determines $L(s)$ uniquely if it is minimum-phase.

Equivalent problem: Find a minimum-phase $L(s)$ which satisfies $S(\omega) = \lvert L(j\omega)\rvert^2$.

A solution exists if

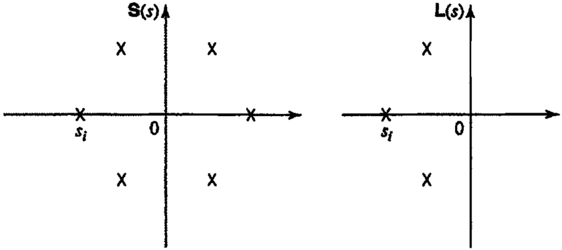
$$\int_{-\infty}^{\infty} \frac{|\log S(\omega)|}{1+\omega^2} d\omega < \infty \quad \text{(Payley-Wiener condition)}$$

Special Case: Rational $S(\omega)$

Let $S(\omega)$ be a rational function of ω^2:

$$S(\omega) = \frac{A(\omega^2)}{B(\omega^2)}, \quad S(s) = \frac{A(-s^2)}{B(-s^2)}.$$

If s is a root of A or B, then $-s$ is also a root.
Furthermore, roots are symmetric with respect to the real axis.



Special Case: Rational $S(\omega)$

Therefore, $S(s)$ has the following representation:

$$S(s) = \frac{N(s)N(-s)}{D(s)D(-s)},$$

where $N(s)$ and $D(s)$ are polynomials of s whose roots are all in the left-half-plane (including $j\omega$ axis). Then we simply have

$$L(s) = \frac{N(s)}{D(s)}.$$

Example. For $S(\omega) = \frac{49+25\omega^2}{(1+\omega^2)(10+\omega^2)}$ we have

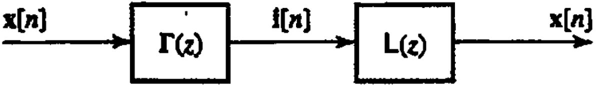
$$S(s) = \frac{49-25s^2}{(1-s^2)(9-s^2)} = \underbrace{\frac{7+5s}{(1+s)(3+s)}}_{L(s)} \cdot \frac{7-5s}{(1-s)(3-s)}.$$

Discrete Time Processes

Let a real WSS digital process $x[n]$ be given.

Problem: Find a *minimum-phase* system $L(z)$ such that $x[n]$ is the output of the system if a *white noise* $i[n]$ is the input.

> A system $L(z)$ is said to be *minimum-phase* if
>
> - it is causal and its impulse response $l[n]$ has finite energy ($\sum_{k=0}^{\infty} l^2[k] < \infty$);
> - its inverse $\Gamma(z) = \frac{1}{L(z)}$ is causal and its impulse response has finite energy.
>
> In other words, $L(z)$ and $\Gamma(z)$ must both be analytic in the exterior $|z| > 1$ of the unit circle.



Discrete Time Processes

Similar to the continuous-time case, while a solution for $L(z)$ doesn't always exist, it is *unique* when it does.

$$i[n] = \sum_{k=0}^{\infty} \gamma(k)x(n-k), \quad R_{ii}[m] = \delta[m]$$

$$x[n] = \sum_{k=0}^{\infty} l[k]i[n-k], \quad E\{x^2[n]\} = \sum_{k=0}^{\infty} l^2[k] < \infty$$

which means

$$S(z) = L(z)L(z^{-1}), \quad S(e^{j\omega}) = |L(e^{j\omega})|^2$$

which determines $L(z)$ uniquely if it is minimum-phase.

Equivalent problem: Find a minimum-phase $L(z)$ which satisfies $S(e^{j\omega}) = |L(e^{j\omega})|^2$.

A solution exists if

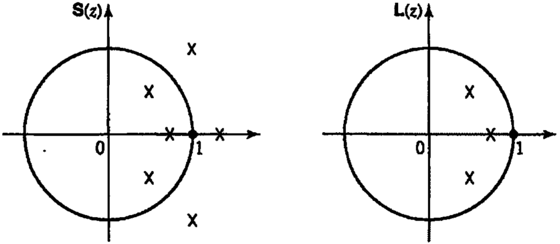
$$\int_{-\pi}^{\pi} |\log S(\omega)| d\omega < \infty \quad \text{(Payley-Wiener condition)}$$

Special Case: Rational $S(e^{j\omega})$

Let $S(e^{j\omega})$ be a function of $\cos\omega = (e^{j\omega} + e^{-j\omega})/2$, which means that $S(z)$ is a function of $z + z^{-1}$:

$$S(z) = \frac{A(z + z^{-1})}{B(z + z^{-1})}.$$

If z is a root of A or B, then z^{-1} is also a root.



Special Case: Rational $S(e^{j\omega})$

Therefore, $S(z)$ has the following representation:

$$S(z) = \frac{N(z)N(z^{-1})}{D(z)D(z^{-1})},$$

where $N(z)$ and $D(z)$ are polynomials of z whose roots are all in or on the unit circle. Then we simply have

$$L(z) = \frac{N(z)}{D(z)}.$$

Example. For $S(z) = \frac{5-2(z+z^{-1})}{10-3(z+z^{-1})}$ we have

$$S(z) = \frac{5-2(z+z^{-1})}{10-3(z+z^{-1})} = \underbrace{\frac{2z-1}{3z-1}}_{L(z)} \cdot \frac{2z^{-1}-1}{3z^{-1}-1}.$$

Finite-Order Systems and State Variables

Continuous Time Case

Linear Time-Invariant Real Causal Systems:

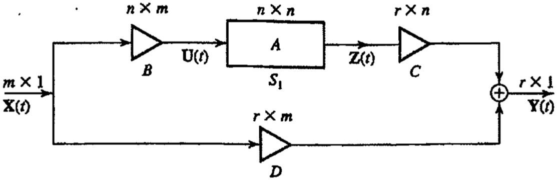
Input vector $X(t) = [x_1(t) \ \ldots \ x_m(t)]^T$; m inputs $x_i(t)$

Output vector $Y(t) = [y_1(t) \ \ldots \ y_r(t)]^T$; r outputs $y_i(t)$

State vector $Z(t) = [z_1(t) \ \ldots \ z_n(t)]^T$; n state variables $z_i(t)$

$$\begin{cases} \dot{Z}(t) &= AZ(t) + BX(t) \\ Y(t) &= CZ(t) + DX(t) \end{cases}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{r\times n}, D \in \mathbb{R}^{r\times m}$.



Continuous Time Case

Let $H(t)$ be the impulse response matrix of the entire system (with input X and output Y). Thus, the response $Y(t)$ to the zero-state ($X(t) = 0,\ t \leq 0$) is

$$Y(t) \ = \int_0^\infty H(\alpha)X(t-\alpha)d\alpha.$$

To determine $H(t)$, let $\Phi(t)$ be the impulse response matrix of the subsystem S_1 (with input $U = BX$ and output Z). We have

$$Z(t) \ = \int_0^\infty \Phi(\alpha)BX(t-\alpha)d\alpha.$$

Therefore,

$$Y(t) \ = \ CZ(t) + DX(t) \ = \int_0^\infty C\Phi(\alpha)BX(t-\alpha)d\alpha + DX(t),$$

or equivalently

$$Y(t) \ = \int_0^\infty \underbrace{[C\Phi(\alpha)B + D\delta(\alpha)]}_{H(\alpha)} X(t-\alpha)d\alpha.$$

Continuous Time Case

Hence

$$\boxed{H(t) \;=\; C\Phi(t)B + D\delta(t)}$$

What is $\Phi(t)$ though? Recall that it is the impulse response of S_1 ($\dot{Z} = AZ + U$). Thus we have

$$\dot{\Phi}(t) \;=\; A\Phi(t) + \delta(t)I_n$$

which implies $s\Phi(s) = A\Phi(s) + I_n$ or $\Phi(s) = (sI_n - A)^{-1}$. Consequently,

$$\boxed{\Phi(t) \;=\; e^{At}}$$

Hence

$$\boxed{H(t) \;=\; Ce^{At}B + \delta(t)D \quad \text{and} \quad H(s) \;=\; C(sI_n - A)^{-1}B + D}$$

Continuous Time Case

Special case: $B = C = I_n$ and $D = 0$, which means that $Y(t) = Z(t)$ and

$$H(s) \;=\; (sI_n - A)^{-1} \qquad \dot{Y}(t) - AY(t) \;=\; X(t).$$

In this case,

$$S_{xy}(s) \;=\; S_{xx}(s)(-sI_n - A)^{-1}$$

$$S_{yy}(s) \;=\; (sI_n - A^T)^{-1}S_{xy}(s) \;=\; (sI_n - A^T)^{-1}S_{xx}(s)(-sI_n - A)^{-1}$$

Continuous Time Case

Differential Equations:

$$y^{(n)}(t) + a_1 y^{(n-1)}(t) + \cdots + a_n y(t) \;=\; x(t),$$

given initial conditions $y(0), \dot{y}(0), \ldots, y^{(n-1)}(0)$.

For this system we simply have

$$H(s) \;=\; \frac{1}{s^n + a_1 s^{n-1} + \cdots + a_n}.$$

This is in fact a special SISO case of the general LTI system presented previously. A *realization* (CCF) of this system in the state space form yields

$$A \;=\; \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ -a_n & -a_{n-1} & \cdots & -a_1 \end{bmatrix} \quad B \;=\; \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad C \;=\; \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T \quad D \;=\; 0.$$

Continuous Time Case

Finite-Order Processes:

A process $x(t)$ is of *finite-order* if its innovation filter $L(s)$ is a rational function of s,

$$L(s) = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_m}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{N(s)}{D(s)},$$

where $N(s)$ and $D(s)$ are polynomials of s whose roots are all in the left-half-plane.

Recall that $x(t)$ is the response of the filter $L(s)$ to the white noise input $i(t)$, i.e.,

$$x^{(n)}(t) + a_1 x^{(n-1)}(t) + \cdots + a_n x(t) = b_0 i^{(m)}(t) + \cdots + b_m i(t).$$

Multiply by $x(t-\tau)$, $\tau > 0$, and take the expected value to obtain

$$\boxed{R^{(n)}(\tau) + a_1 R^{(n-1)}(\tau) + \cdots + a_n R(\tau) = 0}$$

Here we used the fact that for $\tau > 0$, $x(t)$ is orthogonal to $i(t+\tau)$ as they are independent due to causality.

Continuous Time Case

Solving the differential equation, assuming $D(s)$ has only simple roots, we arrive at

$$R(\tau) = \sum_{i=1}^{n} \alpha_i e^{s_i \tau}, \quad \tau > 0,$$

where coefficients α_i are determined from the initial value theorem.

Alternatively, one could employ partial fraction expansion and write

$$S(s) = \underbrace{\sum_{i=1}^{n} \frac{\alpha_i}{s - s_i}}_{\text{causal}} + \underbrace{\sum_{i=1}^{n} \frac{\alpha_i}{-s - s_i}}_{\text{non-causal}},$$

and take the $\mathcal{L}^{-1}$ of the causal part to find $R^+(\tau) \triangleq R(\tau)U(\tau)$.

Similarly, the $\mathcal{L}^{-1}$ of the non-causal part gives $R^-(\tau) \triangleq R(\tau)U(-\tau)$.

Then we have $\boxed{R(\tau) = R^+(|\tau|) = \sum_{i=1}^{n} \alpha_i e^{s_i|\tau|}}$

Continuous Time Case

Example: For $L(s) = \frac{1}{s+\alpha}$, we have

$$S(s) = \frac{1}{(s+\alpha)(-s+\alpha)} = \underbrace{\frac{1/2\alpha}{s+\alpha}}_{\text{causal}} + \underbrace{\frac{1/2\alpha}{-s+\alpha}}_{\text{non-causal}},$$

which results in

$$R(\tau) = (1/2\alpha)e^{-\alpha|\tau|}.$$

Continuous Time Case

Example: Consider the following differential equation:

$$\ddot{x}(t) + 3\dot{x}(t) + 2x(t) \;=\; i(t) \qquad R_{ii}(\tau) \;=\; \delta(\tau).$$

Thus, $L(s) = 1/(s^2 + 3s + 2)$ and

$$S(s) \;=\; \frac{1}{(s^2 + 3s + 2)(s^2 - 3s + 2)} \;=\; \underbrace{\frac{s/12 + 1/4}{s^2 + 3s + 2}}_{\text{causal}} + \underbrace{\frac{-s/12 + 1/4}{s^2 - 3s + 2}}_{\text{non-causal}}.$$

The causal part can be further broken down to $\frac{1/6}{s+1} + \frac{-1/12}{s+2}$. Hence

$$R(\tau) \;=\; \frac{1}{6}e^{-|\tau|} - \frac{1}{12}e^{-2|\tau|}.$$

Discrete Time Case

State equation in discrete time are represented by

$$\begin{aligned} Z[k+1] &= AZ[k] + BX[k] \\ Y[k] &= CZ[k] + DX[k] \end{aligned}$$

Analysis for the continuous time case can easily be carried out for the discrete time case as well to obtain

$$\boxed{H(z) = C(zI_n - A)^{-1}B + D}$$

or equivalently

$$H[k] = C\Phi[k]B + \delta[k]D, \quad k \geq 0$$

Discrete Time Case

Finite-Order Processes:

Let $L(z)$ be the innovation filter of a given finite-order process $x[n]$ with spectrum $S(z)$. Then $L(z)$ can be written as

$$L(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}.$$

Since $x(t)$ is the response of the system $L(z)$ to the white noise $i(t)$,

$$x[n] + a_1 x[n-1] + \cdots + a_N[n-N] = b_0 i[n] + b_1 i[n-1] + \cdots + b_M i[n-M].$$

Remember that

$$S(z) = L(z)L(z^{-1}) \qquad L(z) = \sum_{n=0}^{\infty} l[n] z^{-n}.$$

Also notice that the autocorrelation $R[m]$ of $x[n]$ can be obtained from $L(z)$ from

$$R[m] = l[m] * l[-m] = \sum_{k=0}^{\infty} l[|m|+k] l[k].$$

Discrete Time Case

To obtain $l[n]$, one could take the $\mathcal{Z}^{-1}$ of $L(z)$ using partial fraction expansion:

$$L(z) = \sum_{i=1}^{N} \frac{\gamma_i}{1 - z_i z^{-1}} \Rightarrow l[n] = \sum_{i=1}^{N} \gamma_i z_i^n U[n].$$

Also, $R[m]$ can be derived from the partial fraction expansion of $S(z)$:

$$S(z) = \sum_{i=1}^{N} \frac{\alpha_i}{1 - z_i z^{-1}} + \sum_{i=1}^{N} \frac{\alpha_i}{1 - z_i z} \Rightarrow R[m] = \sum_{i=1}^{N} \alpha_i z_i^{|m|}.$$

Above we assumed that all roots of $D(z)$ are simple.
Note that

$$\alpha_i = \left[(1 - z_i z^{-1}) S(z)\right]\Big|_{z=z_i} = \left[(1 - z_i z^{-1}) L(z)\right]\Big|_{z=z_i} L(z_i^{-1}) = \gamma_i L(z_i^{-1})$$

$$\boxed{\alpha_i = \gamma_i L(z_i^{-1})}$$

Discrete Time Case

Example: For $x[n] - ax[n-1] = i[n]$, we have

$$D(z) = 1 - az^{-1}.$$

Thus, $z_1 = a$ and $R[m] = \alpha_1 a^{|m|}$, where

$$\alpha_1 = \gamma_1 L(a^{-1}) = \frac{1}{1-a^2}.$$

We next study how the coefficients of $L(z)$ are related to the sequence R[m].

Discrete Time Case

Special Case - Autoregressive Processes:
A process is called *autoregressive* (AR) if

$$L(z) = \frac{b_0}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}, \tag{1}$$

which means

$$x[n] + a_1 x[n-1] + \cdots + a_N x[n-N] = b_0 i[n].$$

First, multiply (1) by $i[n]$ and the expected value to obtain

$$\mathbb{E}\{x[n] i[n]\} = b_0.$$

Discrete Time Case

Then, multiply (1) by $x[n-m]$, $m = 0, \ldots, N$ to get

$$\begin{aligned} &R[0] + a_1 R[1] + \cdots + a_N R[n] = b_0^2 \\ &R[1] + a_1 R[0] + \cdots + a_N [n-1] = 0 \\ &\vdots \\ &R[n-1] + a_1 R[n-2] + \cdots a_N R[1] = 0, \end{aligned}$$

which is a system of $N+1$ equation and could be used to determine $R[0], \ldots, R[N-1]$.

Finally, multiply (1) by $x[n-m]$, $m > N$, to get

$$R[m] + a_1 R[m-1] + \cdots + a_N R[m-N] = 0 \qquad m > N.$$

Discrete Time Case

Special Case - Moving Average Processes:
A process is called *moving average* (MA) if

$$L(z) = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M},$$

which means

$$x[n] = b_0 i[n] + b_1 i[n-1] + \ldots + b_M i[n-M].$$

Noting that $l[n] = b_0 \delta[n] + b_1 \delta[n-1] + \cdots + b_M \delta[n-M]$, we have

$$R[m] = \sum_{k=0}^{\infty} l[m+k] l[k] = \sum_{k=0}^{M-m} b_{k+m} b_k \qquad 0 \leq m \leq M.$$

Discrete Time Case

General Case - Autoregressive Moving Average Processes:
A process is called *autoregressive moving average* (ARMA) if

$$x[n] + a_1 x[n-1] + \cdots + a_N x[n-N] \;=\; b_0 i[n] + b_1 i[n-1] + \cdots + b_M i[n-M].$$

Multiplying by $x[n-m]$, $m > M$, and taking the expected value we find

$$R[m] + a_1 R[m-1] + \cdots + a_N R[m-N] \;=\; 0 \qquad m > M.$$

Fourier Series and K-L Expansion

Fourier Series

A process $x(t)$ is said to be *MS periodic* with period T if

$$\mathbb{E}\left\{|x(t+T) - x(t)|^2\right\} = 0 \qquad \forall t.$$

We note that a WSS process is MS periodic with period T if and only if $R(\tau)$ is periodic with period T. (Why?)

Let $T = 2\pi/\omega_0$ and expand $R(\tau)$ into its Fourier series:

$$R(\tau) = \sum_{n=-\infty}^{\infty} \gamma_n e^{jn\omega_0 \tau},$$

where

$$\gamma_n = \frac{1}{T}\int_0^{\infty} R(\tau) e^{-jn\omega_0 \tau} d\tau.$$

We now extend the Fourier series to include MS periodic processes.

Fourier Series

Define

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t},$$

where random variables c_n are derived by

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt.$$

Theorem

Processes $x(t)$ and $\hat{x}(t)$ are equal in the MS sense, i.e.,

$$\mathbb{E}\{|x(t) - \hat{x}(t)|^2\} = 0.$$

Furthermore,

$$\mathbb{E}\{c_n\} = \begin{cases} \eta_x & n = 0 \\ 0 & n \neq 0 \end{cases} \qquad \mathbb{E}\{c_n c_m^*\} = \begin{cases} \gamma_n & n = m \\ 0 & n \neq m \end{cases}$$

Fourier Series

Proof: Showing the part of the theorem involving $\mathbb{E}\{c_n\}$ is quite straightforward. For $\mathbb{E}\{c_n c_m^*\}$, note that the following relations hold:

$$c_n x^*(\alpha) = \frac{1}{T}\int_0^T x(t)x^*(\alpha)e^{-jn\omega_0 t}dt,$$

$$c_n c_m^* = \frac{1}{T}\int_0^T c_n x^*(t)e^{jn\omega_0 t}dt.$$

Taking the expected value and using the linearity of $\mathbb{E}\{.\}$ we conclude

$$\mathbb{E}\{c_n x^*(\alpha)\} = \frac{1}{T}\int_0^T R(t-\alpha)e^{-jn\omega_0 t}dt = \gamma_n e^{-jn\omega_0\alpha},$$

$$\mathbb{E}\{c_n c_m^*\} = \frac{1}{T}\int_0^T \gamma_n e^{-jn\omega_0 t}e^{jm\omega_0 t}dt = \begin{cases}\gamma_n & n=m\\ 0 & n\neq m.\end{cases}$$

Fourier Series

Finally, to show equality of $x(t)$ and $\hat{x}(t)$ in the MS sense, we note that

$$\mathbb{E}\{\hat{x}(t)\hat{x}^*(t)\} = \sum_n \mathbb{E}\{c_n c_n^*\} = \sum_n \gamma_n = R(0),$$

$$\mathbb{E}\{x(t)x^*(t)\} = R(0),$$

$$\mathbb{E}\{\hat{x}(t)x^*(t)\} = \sum_n \mathbb{E}\{c_n x^*(t)\}e^{jn\omega_0 t} = \sum_n \gamma_n e^{-jn\omega_0 t}e^{jn\omega_0 t} = \sum_n \gamma_n = R(0).$$

Expanding $\mathbb{E}\{|x(t)-\hat{x}(t)|^2\}$ and using the equations above, we simply conclude that

$$\mathbb{E}\{|x(t)-\hat{x}(t)|^2\} = 0.$$

Karhunen-Loève Expansion

Functions $e^{jn\omega_0}$ for different values n are orthonormal in the interval $[0, T]$. Furthermore, the resulting variables c_n are orthogonal. This orthogonality would be lost if $x(t)$ were not MS periodic.

We now generalize the Fourier series to Karhunen-Loève expansion dropping the MS periodic property of $x(t)$. In general, define the Karhunen-Loève expansion of $x(t)$ by

$$\hat{x}(t) \triangleq \sum_{n=0}^{\infty} c_n \phi_n(t),$$

where $\phi_n(t)$ –to be determined– are orthonormal over $[0, T]$ and random variables c_n are derived by

$$c_n = \int_0^T x(t)\phi_n^*(t)dt.$$

Orthonormal functions $\phi_n(t)$ will be determined in such a way that functions c_n become orthogonal.

Karhunen-Loève Expansion

To find appropriate functions $\phi_n(t)$, one forms the integral equation

$$\int_0^T R(t_1, t_2)\phi(t_2)dt_2 = \lambda\phi(t_1) \qquad 0 < t_1 < T.$$

The equation above has a sequence of eigenfunctions ϕ_n with corresponding eigenvalues λ_n.

Theorem

If ϕ_n and λ_n are the eigenfunctions and eigenvalues of the integral equation above, we have

$$\mathbb{E}\left\{|x(t) - \hat{x}(t)|^2\right\} = 0,$$

and

$$\mathbb{E}\left\{c_n c_m^*\right\} = \lambda_n \delta[n - m].$$

Karhunen-Loève Expansion

Proof: Recall that

$$c_n = \int_0^T x(t)\phi_n^*(t)dt,$$

$$\int_0^T R(t_1,t_2)\phi(t_2)dt_2 = \lambda\phi(t_1) \qquad 0 < t_1 < T.$$

Therefore, we have

$$\mathbb{E}\{c_n x^*(\alpha)\} = \int_0^T R^*(\alpha,t)\phi_n^*(t)dt = \lambda_n\phi_n^*(\alpha),$$

$$\mathbb{E}\{c_n c_m^*\} = \lambda_n \int_0^T \phi_n^*(t)\phi_m(t)dt = \lambda_n\delta[n-m].$$

Karhunen-Loève Expansion

Furthermore, the following relations hold:

$$\mathbb{E}\left\{\hat{x}(t)\hat{x}^*(t)\right\} = \sum_n \mathbb{E}\left\{c_n c_n^*\right\}|\phi_n(t)|^2 = \sum_n \lambda_n |\phi_n(t)|^2,$$

$$\mathbb{E}\left\{x(t)x^*(t)\right\} = R(t,t) = \sum_n \lambda_n |\phi_n(t)|^2,$$

$$\mathbb{E}\left\{\hat{x}(t)x^*(t)\right\} = \sum_n \lambda_n \phi_n(t)\phi_n^*(t) = \sum_n \lambda_n |\phi_n(t)|^2.$$

Expanding $\mathbb{E}\left\{|x(t) - \hat{x}(t)|^2\right\}$ and using the equations above, we conclude that

$$\mathbb{E}\left\{|x(t) - \hat{x}(t)|^2\right\} = 0.$$