BANDLIMITED PROCESSES AND SAMPLING THEORY

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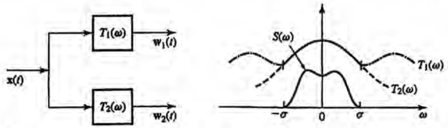
Bandlimited Process

A WSS process is called *bandlimited* (BL) if $\exists\sigma$, $S(\omega) = 0$ if $|\omega| > \sigma$ and $R(0) < \infty$.

Theorem

Assume that

- $x(t)$ *is bandlimited with bound* σ,
- $w_1(t), w_2(t)$ *are the responses of the systems* $T_1(\omega), T_2(\omega)$ *to* $x(t)$, *and*
- $T_1(\omega) = T_2(\omega)$, $\forall\omega$, $|\omega| < \sigma$.



Then we have $w_1(t) = w_2(t)$, $\forall t$.

Proof: $w_1(t) - w_2(t)$ is the response of the system $T_1(\omega) - T_2(\omega)$ to $x(t)$. Thus

$$E\{|w_1(t) - w_2(t)|^2\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)|T_1(\omega) - T_2(\omega)|^2 d\omega$$

which equals to 0 since at least one of $S(\omega)$ and $T_1(\omega) - T_2(\omega)$ is 0 for any ω. Therefore, $w_1(t) = w_2(t)$.

Sampling Expansions

> *Reminder:* Given a deterministic signal $f(t)$
> whose Fourier transform satisfies $F(\omega) = 0$, for $|\omega| > \sigma$,
> $f(t)$ can be expressed in terms of its samples $f(nT)$,
> where $T = \pi/\sigma$ is the Nyquist interval.

Thus, given a bandlimited process $x(t)$ with autocorrelation $R(\tau)$, we can apply this sampling expansion result to $R(\tau)$, which is deterministic, as follows:

$$R(\tau) = \sum_{n=-\infty}^{\infty} R(nT) \frac{\sin \sigma(\tau - nT)}{\sigma(\tau - nT)}.$$

This sampling expansion for deterministic signals can be extended to stochastic processes, according to the following theorem.

Sampling Expansions

Theorem

Given a BL process $x(t)$, we can write

$$x(t+\tau) = \sum_{n=-\infty}^{\infty} x(t+nT)\frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)} \quad \forall t,\tau \quad T=\frac{\pi}{\sigma}$$

Write the Fourier series of $e^{j\omega\tau}$, assuming ω is the variable, in the interval $[-\sigma,\sigma]$. The coefficients of the series are

$$a_n = \frac{1}{2\sigma}\int_{-\sigma}^{\sigma} e^{j\omega\tau}e^{-jnT\omega}d\omega = \frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}.$$

Thus

$$e^{j\omega\tau} = \sum_{n=-\infty}^{\infty} e^{jnT\omega}\frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)} \quad |\omega|\le\sigma.$$

Sampling Expansions

Now, defining

$$T_1(\omega) \triangleq e^{j\omega\tau},$$

$$T_2(\omega) \triangleq \sum_{n=-\infty}^{\infty} e^{jnT\omega} \frac{\sin\sigma(\tau - nT)}{\sigma(\tau - nT)} \qquad |\omega| \leq \sigma,$$

we note that

- $x(t)$ is bandlimited with bound σ,
- $x(t+\tau)$ and $\sum_{n=-\infty}^{\infty} x(t+nT)\frac{\sin\sigma(\tau-nT)}{\sigma(\tau-nT)}$ are the responses of the systems $T_1(\omega)$ and $T_2(\omega)$ to $x(t)$, respectively, and
- $T_1(\omega) = T_2(\omega)$, $\forall\omega$, $|\omega| < \sigma$.

Therefore, according the the previous theorem, we must have

$$x(t+\tau) = \sum_{n=-\infty}^{\infty} x(t+nT)\frac{\sin\sigma(\tau - nT)}{\sigma(\tau - nT)}.$$

Past Samples

A deterministic BL signal cannot be determined from its samples in real time since it would requires samples from future times as well.
This is not the case for BL random processes if a sampling time $T_0 < T = \pi/\sigma$ is used.

Example: Let $x(t)$ be a BL process with σ and define for a fixed n

$$\hat{x}(t) = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} x(t - kT_0).$$

Then $x(t) - \hat{x}(t) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} x(t - kT_0)$, that is the response to $x(t)$ of system

$$H(\omega) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} e^{-jkT_0\omega} = (1 - e^{-j\omega T_0})^n.$$

Thus, $E\{(x(t) - \hat{x}(t))^2\} = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} S(\omega)|H(\omega)|^2 d\omega$, where $\lim_{n\to\infty} |H(\omega)| = 0$ if $T_0 < \frac{T}{3} = \frac{\pi}{3\sigma}$. (Why?) This means that $\hat{x}(t)$ tends to $x(t)$ as n grows.

Past Samples

Theorem

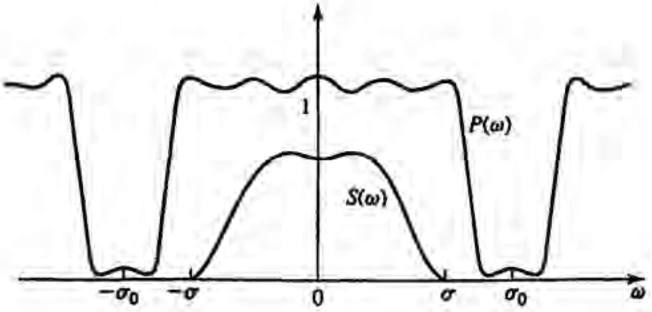
Given $T_0 < T$ and an arbitrary $\epsilon > 0$,
there is a sufficiently large n
and coefficients $a_1, \ldots, a_n$
such that for $\hat{x}(t) \triangleq \sum_{k=1}^{n} a_k e^{-jkT_0\omega}$ we have

$$E\{(x(t) - \hat{x}(t))^2\} < \epsilon.$$

Sketch of Proof: $\hat{x}(t)$ is the response to $x(t)$ of the system

$$P(\omega) = \sum_{k=1}^{n} a_k e^{-jkT_0\omega}.$$

Thus, $E\{(x(t) - \hat{x}(t))^2\} = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} S(\omega)|1 - P(\omega)|^2 d\omega$. Therefore, all it takes is to make $P(\omega)$ close to 1 for all $\omega \in [-\sigma, \sigma]$. Taking into account that $P(\omega)$ is periodic with period π/T_0, such a task is possible if $\pi/T_0 < \sigma$, or equivalently $T_0 < T$.


1
P(ω)
S(ω)
−σ0
−σ
0
σ
σ0
ω

Random Sampling

The goal now is to estimate the Fourier transform $F(\omega)$ of a deterministic signal $f(t)$ using samples of $f(t)$. Consider the estimate $F_*(\omega)$ of $F(\omega)$ defined as

$$F_*(\omega) \triangleq \sum_{n=-\infty}^{\infty} Tf(nT)e^{-jn\omega T}.$$

We can show that

$$F_*(\omega) = \sum_{n=-\infty}^{\infty} F(\omega + 2n\sigma), \qquad \sigma \triangleq \pi/T.$$

This estimate of $F(\omega)$ is satisfactory only if $F(\omega)$ is negligible outside $(-\sigma, \sigma)$. Let us instead of times nT, sample $f(t)$ at times t_i where t_i is a Poisson point process with average density λ.

Theorem

For $P(\omega) \triangleq \frac{1}{\lambda}\sum_i f(t_i)e^{-j\omega t_i}$ *we have*

$$E\{P(\omega)\} = F(\omega) \qquad \sigma^2_{P(\omega)} = \frac{E}{\lambda} \qquad \text{where } E \triangleq \int_{-\infty}^{\infty} f^2(t)dt.$$

Random Sampling

A satisfactory estimate is achieved if $|F(\omega)| \gg (E/\lambda)^{1/2}$.

Example: Let

$$f(t) = \begin{cases} \sum_k c_k e^{j\omega_k t} & |t| < a, \\ 0 & |t| > a. \end{cases}$$

Then we have

$$F(\omega) = \sum_k 2c_k \frac{\sin a(\omega - \omega_k)}{\omega - \omega_k}.$$

We can estimate the energy E by $2a \sum_k |c_k|^2$. Also if a is large, $f(\omega_k)$ is roughly $2ac_k$. Thus, the estimate $P(\omega_k)$ is satisfactory if

$$\sum_i |c_i|^2 \ll 2a\lambda |c_k|^2.$$