CYCLOSTATIONARY PROCESSES

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Definitions and Main Theorems

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Definition: A process $x(t)$ is called strict-sense cyclostationary (SSCS) with period T if statistical properties of $x(t)$ are invariant under a shift of the origin by any integer multiple of T, i.e.,

$$F(x_1, \ldots, x_n; t_1 + mT, \ldots, t_n + mT) = F(x_1, \ldots, x_n; t_1, \ldots, t_n), \ \forall m \in \mathbb{Z}.$$

It is called wide-sense cyclostationary (WSCS) if

$$\eta(t + mT) = \eta(t) \ \text{ and } \ R(t_1 + mT, t_2 + mT) = R(t_1, t_2), \ \forall m \in \mathbb{Z}.$$

Notice that *every SSCS process is also WSCS*, while the converse is not always true.

Theorem

Let $x(t)$ be SSCS, $\Theta = Unif(0, T)$, and $\Theta \perp x$. Then, $\bar{x}(t) \triangleq x(t - \Theta)$ is SSS. Furthermore,

$$\bar{F}(x_1, \ldots, x_n; t_1, \ldots, t_n) = \frac{1}{T} \int_0^T F(x_1, \ldots, x_n; t_1 - \alpha, \ldots, t_n - \alpha) d\alpha.$$

Definitions and Main Theorems

Proof: Define an event A by

$$A \triangleq \{\bar{x}(t_1 + c) \leq x_1, \ldots, \bar{x}(t_n + c) \leq x_n\}.$$

It is sufficient to show that $P(A)$ is independent of c (for the SSS part of the theorem) and is equal to

$$\frac{1}{T} \int_0^T F(x_1, \ldots, x_n; t_1 - \alpha, \ldots, t_n - \alpha) d\alpha.$$

We know that

$$P(A) = \frac{1}{T} \int_0^T P(A|\Theta = \theta) d\theta.$$

On the other hand,

$$P(A|\Theta = \theta) = P\Big(x(t_1 + c - \theta) \leq x_1, \ldots, x(t_n + c - \theta) \leq x_n \,|\theta\Big).$$

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Since $\Theta \perp x$, we have

$$\begin{aligned} P(A|\Theta=\theta) &= P\big(x(t_1+c-\theta)\le x_1,\ldots,x(t_n+c-\theta)\le x_n\big) \\ &= F(x_1,\ldots,x_n;t_1+c-\theta,\ldots,t_n+c-\theta) \\ &= F\big(x_1,\ldots,x_n;t_1-(\theta-c\ mod\ T),\ldots,t_n-(\theta-c\ mod\ T)\big). \end{aligned}$$

Thus

$$\frac{1}{T}\int_0^T P(A|\Theta=\theta)d\theta$$

$$\frac{1}{T}\int_0^T \Big[F\big(x_1,\ldots,x_n;t_1-(\theta-c\ mod\ T),\ldots,t_n-(\theta-c\ mod\ T)\big)\Big]d\theta.$$

Defining $\alpha \triangleq (\theta - c\ mod\ T)$, the integral on the RHS becomes equal to

$$\frac{1}{T}\int_0^T F(x_1,\ldots,x_n;t_1-\alpha,\ldots,t_n-\alpha)d\alpha.$$

Definitions and Main Theorems

Theorem

Let $x(t)$ be WSCS, $\Theta = Unif(0,T)$, and $\Theta \perp x$. Then, $\bar{x}(t) \triangleq x(t-\Theta)$ is WSS with

$$E\{\bar{x}(t)\} = \frac{1}{T}\int_0^T E\{x(t)\}dt$$

and

$$\bar{R}(\tau) = E\{\bar{x}(t+\tau)\bar{x}(t)\} = \frac{1}{T}\int_0^T R(t+\tau,t)dt.$$

Proof: For the mean, since $\Theta \perp x$, we have

$$\begin{aligned} E\{\bar{x}(t)\} &= E\{x(t-\Theta)\} = E\{\eta(t-\Theta)\}\} \\ &= \tfrac{1}{T}\textstyle\int_0^T \eta(t-\theta)d\theta \\ &= \tfrac{1}{T}\textstyle\int_0^T \eta(t)dt. \end{aligned}$$

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For the autocorrelation,

$$\begin{aligned} E\{\bar{x}(t+\tau)\bar{x}(t)\} &= E\Big\{x(t+\tau-\Theta)x(t-\Theta)\Big\} \\ &= E\Big\{R(t+\tau-\Theta, t-\Theta)\Big\} \\ &= \frac{1}{T}\int_0^T R(t+\tau-\theta, t-\theta)d\theta \\ &= \frac{1}{T} \in_0^T R(t+\tau, t)dt. \end{aligned}$$

Pulse-Amplitude Modulation (PAM)

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Let $h(t)$ be a given function with Fourier transform $H(\omega)$, c_n be a stationary sequence of random variables with autocorrelation $R_c[m] = E\{c_{n+m}c_n\}$ and power spectrum

$$S_c(e^{j\omega}) = \sum_{m=-\infty}^{\infty} R_c[m]e^{-jm\omega}.$$

Then, random process $x(t)$ defined by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n h(t - nT)$$

is a cyclostationary process. The process of generating $x(t)$ is called *pulse-amplitude modulation*.

Theorem

The power spectrum $\bar{S}_x(\omega)$ of the shifted process $\bar{x}(t)$ is equal to

$$\bar{S}_x(\omega) = \sum_{m=-\infty}^{\infty} \frac{1}{T} S_c(e^{j\omega})|H(\omega)|^2.$$

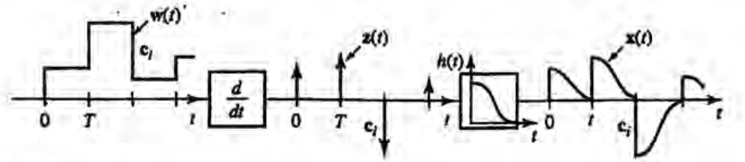
Pulse-Amplitude Modulation (PAM)

Proof: By definition, it is clear that $x(t) = z(t) * h(t)$ where $z(t)$ is the impulse train generated by c_n as

$$z(t) = \sum_{n=-\infty}^{\infty} c_n \delta(t - nT).$$

Let cyclostationary process $w(t)$ be defined by

$$w(t) \triangleq \int_{-\infty}^{t} z(t')dt' = \sum_{n=-\infty}^{\infty} c_n U(t - nT)$$



Pulse-Amplitude Modulation (PAM)

We have

$$R_w(t_1,t_2) = \sum_{n=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} R_c[n-r]U(t_1-nT)U(t_2-rT).$$

Thus

$$R_z(t_1,t_2) = \sum_{n=-\infty}^{\infty} \sum_{r=-\infty}^{\infty} R_c[n-r]\delta(t_1-nT)\delta(t_2-rT).$$

Therefore

$$R_z(t+\tau,t) = \sum_{m=-\infty}^{\infty} R_c[m] \sum_{r=-\infty}^{\infty} \delta\big(t+\tau-(m+r)T\big)\delta(t-rT).$$

We next find $\bar{R}_z(\tau)$ and its Fourier transform $\bar{S}_z(\omega)$, for the shifted process $\bar{z}(t)$.

Pulse-Amplitude Modulation (PAM)

From the theorem we had earlier, and a bit of computations, we obtain

$$\bar{R}_z(\tau) = \frac{1}{T} \sum_{m=-\infty}^{\infty} R_c[m]\delta(\tau - mT),$$

which results in

$$\bar{S}_z(\omega) = \frac{1}{T} \sum_{m=-\infty}^{\infty} R_c[m]e^{-jmT\omega} = \frac{1}{T} S_c(e^{j\omega}).$$

Now, since $x = z * h$, from linearity we get $\bar{x} = \bar{z} * h$. Thus

$$\bar{S}_x = \bar{S}_z H H^* = \bar{S}_z |H|^2,$$

which proves the theorem.

Pulse-Amplitude Modulation (PAM)

Example: Let $h(t)$ be a pulse

$$h(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

and c_n be a white-noise process taking values ± 1 with equal probability. Notice that $R_c[m] = \delta[m]$.
Now the process $x(t) = \sum_{n=-\infty}^{\infty} c_n h(t - nT)$ is called *binary transmission*. It is SSCS and for the shifted process $\bar{x}(t)$ we have $\bar{S}_x(\omega) = \frac{4\sin^2(\omega T/2)}{t\omega^2}$.