OBSERVABILITY, DUALITY, AND MINIMALITY

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Overview

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An Overview
Definition
The LTV system above is said to be **observable at time** $t_0$ if there exists some finite time $t_1$ such that for any $x_0 \in \mathbb{R}^n$, knowledge of the input $u(t)$ and the output $y(t)$ for $t_0 \leq t \leq t_1$ suffices to determine $x_0$.

Remember that $x(t) = \phi(t, t_0)x_0 + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau$, that means

$$y(t) = C(t)\phi(t, t_0)x_0 + C(t) \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

▶ Observability at $t_0$ only depends on $C(t)$ and $\phi(t, t_0)$ for $t \geq t_0$. 
LTI Case
LTI Case

\[
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}
\]

Observability Matrix:

\[
\mathcal{O} := \begin{bmatrix}
C \\
CA \\
CA^2 \\
\vdots \\
CA^{n-1}
\end{bmatrix}
\]

Theorem

The LTI system above is **observable** (at any \( t_0 \)) if and only if the observability matrix \( \mathcal{O} \) is full-column-rank.

- If the LTI system above is observable, *the pair* \((A, C)\) *is said to be observable*.
- The kernel of \( \mathcal{O} \) is referred to as the *unobservable subspace*. 
General LTV Case
General LTV Case

\[\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{align*}\]

**Theorem**

An LTV system is observable at $t_0$ if and only if the observability Gramian $H(t_1, t_0)$ is nonsingular for some finite $t_1 > t_0$ where

\[H(t_1, t_0) := \int_{t_0}^{t_1} \phi(\tau, t_0)C^T(\tau)C(\tau)\phi(\tau, t_0) d\tau.\]
Duality
Controllability and observability are dual concepts.

Model I:
\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{align*}
\]

Dual of Model I:
\[
\begin{align*}
\dot{x}(t) &= -A^T(t)x(t) + C^T(t)u(t) \\
y(t) &= B^T(t)x(t) + D^T(t)u(t)
\end{align*}
\]

Model I is controllable/observable if and only if its dual is observable/controllable.

Controllability tests can now be transformed to observability tests. For instance

Theorem

An LTI system is observable if and only if \([sI - A^T | C^T]\) is full-row-rank for any \(s \in \mathbb{C}\).

▶ unobservable mode \(\lambda\): \([\lambda I - A^T | C^T]\) is not full-row-rank.
Kalman Canonical Forms
\[
\begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\quad \overset{x = Px}{\implies} \quad
\begin{align*}
\dot{x} &= \tilde{A}\bar{x} + \tilde{B}u \\
y &= \tilde{C}\bar{x} + \tilde{D}u
\end{align*}

\tilde{A} = PAP^{-1}, \quad \tilde{B} = PB, \quad \tilde{C} = CP^{-1}, \quad \tilde{D} = D
Kalman Controllability Canonical Form (KCCF)

Let the controllability matrix $C$ has rank $n_1$.

$$P := \begin{bmatrix} n_1 \text{ linearly independent columns of } C & (n - n_1)\text{ vectors to make } P \text{ non-singular} \end{bmatrix}^{-1}$$

Then

$$\bar{A} = \begin{bmatrix} A_c & \ast \\ 0 & A_c^* \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_c \\ 0 \end{bmatrix}$$

where $(A_c, B_c)$ is controllable.
We can find $P$ such that $\bar{A} = PAP^{-1}$ and $\bar{C} = CP^{-1}$ have the following forms:

$$\bar{A} = \begin{bmatrix} A_o & \mathbf{0} \\ \ast & A_\bar{o} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_o & \mathbf{0} \end{bmatrix}$$

where $(A_o, C_o)$ is observable.
Minimality
Minimality

Remember *realizations*?

\[ G(s) : \text{proper rational function of } s \]

\( (A, B, C, D) \) is a realization of \( G(s) \) if

\[ G(s) = C(sI - A)^{-1}B + D. \]

**Definition**

A realization of \( G(s) \) is said to be **minimal** if it involves a minimal number of state variables.

**Theorem**

A realization of \( G(s) \) is **minimal** if and only if it is *both* controllable and observable.